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Notes for Starlikeness Conditions of Analytic Functions

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1. Introduction

Let A denote the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$, with

$$(1.1) \quad f(0)=0 \text{ and } f'(0)=1.$$

And let S^* be the usual class of starlike functions in D , i. e.

$$S^*=\left\{f \in A: \operatorname{Re} \frac{z f'(z)}{f(z)} > 0, z \in D\right\}.$$

On the starlikeness condition for functions of A , we now investigate the following well known problems:

Problem 1. Find the maximum value of r for which $f \in A$ and

$$(1.2) \quad |f'(z)-1|<1, z \in D$$

imply that f is starlike in

$$(1.3) \quad D_r = \{z: |z| < r\}.$$

Problem 2. Find the maximum value of r for which $f \in A$ and

$$(1.4) \quad |f'(z) - 1| < r, \quad z \in D$$

imply that $f \in S^*$.

We denote by r_1 and r_2 the maximum value of the first problem and of the second problem, respectively.

For r_1 , T. MacGregor [1] showed that

$$r_1 \geq 2/\sqrt{5} = 0.894\ldots$$

Recently, M. Nunokawa in [3] proved that $r_1 > 0.926\ldots$, and in [4] Nunokawa et al improved, that is $r_1 > 0.933\ldots$. On the other hand, P. Mocanu [2] showed that the function

$$(1.5) \quad f(z) = -2z + z^2 + 6 \log \frac{2+z}{2}$$

belongs to A and satisfies the condition (1.2), but does not belong to S^* . Therefore $r_1 < 1$. For r_2 , in the same article Mocanu also proved that

$$r_2 \geq 2/\sqrt{5} = 0.894\ldots$$

In this short paper, modifying the Mocanu's example, we show that

$$(1.6) \quad r_1 < 0.9982$$

and

$$(1.7) \quad r_2 < 0.9962.$$

2. Examples

Example 1. Let

$$(2.1) \quad f(z) = (2 - a^2)z + \frac{a}{2}z^2 + a(a^2 - 1)\log \frac{z+a}{a}, \quad z \in D,$$

where $\log 1 = 0$. Since

$$(2.2) \quad f'(z) = 2 - a^2 + az + \frac{a(a^2 - 1)}{z+a} = 1 + z \frac{az+1}{z+a},$$

we easily deduce

$$f(0) = f'(0) - 1 = 0$$

and

$$|f'(z) - 1| < 1, \quad z \in D.$$

If we let

$$f(z) = s + it, \quad zf'(z) = u + iv$$

and $z = re^{i\theta}$, then we have

$$s = (2 - a^2)r \cos \theta + \frac{a}{2}r^2 \cos 2\theta + \frac{a(a^2 - 1)}{2} \log(r^2 + a^2 + 2ar \cos \theta) - a(a^2 - 1) \log a,$$

$$t = (2 - a^2)r \sin \theta + \frac{a}{2}r^2 \sin 2\theta + a(a^2 - 1) \tan^{-1} \frac{r \sin \theta}{a + r \cos \theta},$$

$$u = (2 - a^2)r \cos \theta + ar^2 \cos 2\theta + a(a^2 - 1) - \frac{a^2(a^2 - 1)(r \cos \theta + a)}{r^2 + a^2 + 2ar \cos \theta}$$

and

$$v = (2 - a^2) r \sin \theta + a r^2 \sin 2\theta + \frac{a^2(a^2 - 1) r \sin \theta}{r^2 + a^2 + 2a r \cos \theta}.$$

And it is evident that $\operatorname{Re}[z f' / f] > 0$ if and only if

$$(2.3) \quad s u + t v > 0.$$

But by putting

$$(2.4) \quad a = 1.1, \quad r = 0.9982, \quad \theta = 2.743,$$

we deduce $s u + t v = -9.13 \times 10^{-5} < 0$, which shows that f is not starlike in D_r , i.e. $r_1 < 0.9982$.

Example 2. Let

$$(2.5) \quad g(z) = \left(1 - \frac{b}{a}\right) z + \frac{b}{2(a^2 - 1)} z^2 + b \log(z + a) - b \log a.$$

Since

$$(2.6) \quad g'(z) = 1 + \frac{b}{a(a^2 - 1)} \cdot \frac{z(a z + 1)}{z + a},$$

evidently $g \in A$, and

$$(2.7) \quad |g'(z) - 1| < k = \left| \frac{b}{a(a^2 - 1)} \right|, \quad z \in D.$$

Let

$$g(z) = s + i t, \quad z f'(z) = u + i v$$

and $z = r e^{i\theta}$. Then we deduce

$$s = \left(1 - \frac{b}{a}\right) r \cos \theta + \frac{b}{2(a^2 - 1)} r^2 \cos 2\theta + \frac{b}{2} \log(r^2 + a^2 + 2a r \cos \theta) - b \log a,$$

$$t = \left(1 - \frac{b}{a}\right) r \sin \theta + \frac{b}{2(a^2 - 1)} r^2 \sin 2\theta + b \tan^{-1} \frac{r \sin \theta}{a + r \cos \theta},$$

$$u = \left(1 - \frac{b}{a}\right) r \cos \theta + \frac{b}{a^2 - 1} r^2 \cos 2\theta + \frac{b(r^2 + a r \cos \theta)}{r^2 + a^2 + 2a r \cos \theta},$$

$$v = \left(1 - \frac{b}{a}\right) r \sin \theta + \frac{b}{a^2 - 1} r^2 \sin 2\theta + \frac{a b r \sin \theta}{r^2 + a^2 + 2a r \cos \theta}.$$

Substituting

$$(2.8) \quad a = 1.1, \quad b = 0.2301, \quad r = 1, \quad \theta = 2.743$$

we obtain $su + tv = -4.36 \times 10^{-5} < 0$, and

$$(2.9) \quad k = 0.9961038 \dots,$$

which yield $r_2 < 0.9962$.

References

- [1] T. H. MacGregor, A Class of univalent functions, Proc. Amer. Math. Soc., 15(1964), 311-317.
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